Problem Sheet 7

Problems in Part A will be discussed in class. Problems in Part B come with solutions and should be tried at home.

Part A

(7.1) Show that if $f(\boldsymbol{x})$ is a convex function, then the set $\{\boldsymbol{x} : f(\boldsymbol{x}) \leq 0\}$ is a convex set. Conclude that the feasible set

$$\mathcal{C} = \{\boldsymbol{x} \in \mathbb{R}^n : g_1(\boldsymbol{x}) \leq 0, \dots, g_m(\boldsymbol{x}) \leq 0, h_1(\boldsymbol{x}) = 0, \dots, h_\ell(\boldsymbol{x}) = 0\},$$

with g_i convex and h_j linear, is a convex set.

(7.2) Given a constrained optimization problem with equality constraints

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) \\ \text{subject to} \quad & g_1(\boldsymbol{x}) = \cdots = g_m(\boldsymbol{x}) = 0, \end{aligned}$$

the **Lagrangian** function is defined as the function in $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{\lambda} \in \mathbb{R}^m$,

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = f(\boldsymbol{x}) - \sum_{i=1}^m \lambda_i g_i(\boldsymbol{x}) = f(\boldsymbol{x}) - \langle \boldsymbol{\lambda}, \boldsymbol{g}(\boldsymbol{x}) \rangle.$$

A point $\boldsymbol{x}$ is a local minimimum of the equality constrained optimization problem, if there exist **Lagrange multipliers** $\boldsymbol{\lambda} \in \mathbb{R}^m$ such that the Lagrangian satisfies $\nabla \mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = \boldsymbol{0}$, where the gradient is with respect to both sets of variables $(\boldsymbol{x}, \boldsymbol{\lambda})$. If f is convex and the g_i linear, this is a necessary and sufficient condition for a global minimum.

Use the method of Lagrange multipliers to find a closed-form solution for the minimum of an equality constrained quadratic optimization problem

$$\text{minimize} \quad \frac{1}{2} \boldsymbol{x}^\top \boldsymbol{Q} \boldsymbol{x} \quad \text{subject to} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}.$$

(7.3) Consider the optimization problem

$$\text{minimize} \quad x_1^2 + x_2^2 \qquad \text{subject to} \quad \frac{x_1}{1 + x_2^2} \leq 0, \ (x_1 + x_2)^2 = 0. \tag{1}$$

Show that this problem is not a convex optimization problem. Derive a convex optimization problem that has the same solution as (1)

(7.4) Given a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$, formulate the first-order optimality conditions for the problem

$$f(\boldsymbol{x}) = -\sum_{i=1}^m \log(b_i - \boldsymbol{a}_i^\top \boldsymbol{x}),$$

with the constraints $\boldsymbol{A}\boldsymbol{x} + \boldsymbol{s} = \boldsymbol{b}$ and $\boldsymbol{s} > \boldsymbol{0}$. Compute the Lagrange dual.

Part B

(7.5) Consider the following portfolio optimization problem.

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{x}^\top \Sigma \boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{r}^\top \boldsymbol{x} = \mu \\ & \boldsymbol{e}^\top \boldsymbol{x} = 1. \end{aligned} \tag{1}$$

where

- $\boldsymbol{\Sigma} \in \mathbb{R}^{n\times n}$ is a positive semidefinite symmetric matrix;
- $\boldsymbol{e} = (1, \dots, 1)^\top$;
- $\boldsymbol{r} \in \mathbb{R}^n$ is a vectors of estimated returns.

The interpretation is that $\boldsymbol{\Sigma}$ is an estimated covariance matrix, and the goal is to find an investment strategy that minimizes the risk for a given return level. Using the method of **Lagrange multipliers**, show that the solution is characterized by:

$$x = \frac{1}{ac-b^2}\left(c\boldsymbol{\Sigma}^{-1}r - b\boldsymbol{\Sigma}^{-1}e\right) + \mu \cdot \left(a\boldsymbol{\Sigma}^{-1}e - b\boldsymbol{\Sigma}^{-1}r\right),$$

where $a = e^\top \boldsymbol{\Sigma}^{-1} e$, $b = e^\top \boldsymbol{\Sigma}^{-1} r$ and $c = r^{-1}\boldsymbol{\Sigma}^{-1} r$.

Given the covariance matrix and expected returns as follows,

$$\boldsymbol{r} = \begin{pmatrix} 14 \\ 12 \\ 15 \\ 7 \end{pmatrix}, \quad \boldsymbol{\Sigma} = \begin{pmatrix} 185 & 86.5 & 80 & 20 \\ 86.5 & 196 & 76 & 13.5 \\ 80 & 76 & 411 & -19 \\ 20 & 13.5 & -19 & 25 \end{pmatrix},$$

Compute the *efficient frontier*, i.e., the plot that relates the solution of (1) to target returns μ for μ varying between 5 and 35.

Repeat the same exercise, but this time with the additional constraint $\boldsymbol{x} \geq 0$. You can use CVXPY for that. Give an interpretation of this additional constraint.

(7.6) Consider the *Boolean* optimization problem

$$\begin{aligned} \text{minimize} \quad & \langle \boldsymbol{c}, \boldsymbol{x} \rangle \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \\ & x_i \in \{0, 1\}, \; 1 \leq i \leq n. \end{aligned}$$

This problem requires the x_i to have integer values, and falls outside the scope of continuous optimization. Show that the problem is equivalent to

$$\begin{aligned} \text{minimize} \quad & \langle \boldsymbol{c}, \boldsymbol{x} \rangle \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \\ & x_i(1 - x_i) = 0, \; 1 \leq i \leq n. \end{aligned}$$

While this problem is not convex (the equality constraints are quadratic), we can still formulate the Lagrange dual to this problem, whose optimal value gives a lower bound. Show that the Lagrange dual is a convex optimization problem, thus giving a way to *approximate* the solution of the discrete problem by solving a convex optimization problem.