Solutions to Part A of Problem Sheet 8

Solution (8.1) We write the optimization problem slightly different as

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) + \frac{1}{t}\varphi(\boldsymbol{x}) \\ \text{subject to} \quad & f_i(\boldsymbol{x}) \leq \mathbf{0} \\ & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}. \end{aligned}$$

Using the fact that the gradient of $\varphi(\boldsymbol{x})$ is given by

$$\nabla_{\boldsymbol{x}}\varphi(\boldsymbol{x}) = -\sum_{i=1}^m \frac{\nabla f_i(\boldsymbol{x})}{f_i(\boldsymbol{x})},$$

we can derive the KKT conditions for this problem as

$$\begin{aligned} \boldsymbol{f}(\boldsymbol{x}) &\leq \mathbf{0} \\ \boldsymbol{A}\boldsymbol{x} &= \boldsymbol{b} \\ \tilde{\boldsymbol{\lambda}} &\geq \mathbf{0} \\ \tilde{\lambda}_i f_i(\boldsymbol{x}) &= 0,\ 1 \leq i \leq m \\ \nabla_{\boldsymbol{x}} f(\boldsymbol{x}^*) - \frac{1}{t}\sum_{i=1}^m \frac{1}{f_i(\boldsymbol{x})}\nabla f_i(\boldsymbol{x}) + \sum_{i=1}^m \tilde{\lambda}_i \nabla f_i(\boldsymbol{x}) + \boldsymbol{A}^\top \boldsymbol{\mu} &= \mathbf{0}, \end{aligned}$$

We now join the coefficients of $\nabla f_i(\boldsymbol{x})$ in the last line and set

$$\lambda_i := \tilde{\lambda}_i - \frac{1}{t f_i(\boldsymbol{x})}.$$

The last equation then becomes

$$\nabla_{\boldsymbol{x}} f(\boldsymbol{x}^*) + \sum_{i=1}^m \lambda_i \nabla f_i(\boldsymbol{x}) + \sum_{i=1}^m + \boldsymbol{A}^\top \boldsymbol{\mu} = \mathbf{0}.$$

These new multipliers also satisfy

$$\lambda_i f_i(\boldsymbol{x}) = -\frac{1}{t}.$$

Rewriting the KKT conditions for the barrier problem in terms of the λ_i gives the modified KKT conditions.

Solution (8.2) We use $(x-1)(x+1) = x^2 - 1$. The Lagrangian function is

$$\mathcal{L}(x, \lambda) = x^2 - 1 + \lambda(x-1)(x+4).$$

The term $(x-1)(x+4)$ has a minimum at $x=-3/2$. By setting the derivative of the Lagrangian to zero, we get the critical point

$$x = -\frac{3\lambda}{2(1+\lambda)},$$

which is a minimum for $\lambda > -1$. If $\lambda \leq -1$, then one easily checks that the Lagrangian is unbounded from below (we have the x^2 term with a non-positive coefficient). It follows that for $\lambda > -1$, the Lagrange dual is given by

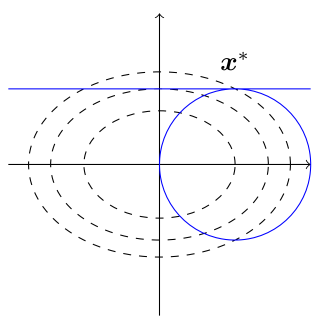
$$g(\lambda) = \inf_x \mathcal{L}(x,\lambda) = \frac{9\lambda^2}{4(1+\lambda)^2} - 1 + \lambda\left(-\frac{3\lambda}{2(1+\lambda)} - 1\right)\left(-\frac{3\lambda}{2(1+\lambda)} + 4\right).$$

For $\lambda \leq -1$, $g(\lambda) = -\infty$. The Lagrange dual problem is

$$\text{maximize}_{\lambda\geq 0} \frac{9\lambda^2}{4(1+\lambda)^2} - 1 + \lambda\left(-\frac{3\lambda}{2(1+\lambda)} - 1\right)\left(-\frac{3\lambda}{2(1+\lambda)} + 4\right)$$

Strong duality holds, as Slater's conditions are satisfied (the problem is convex and has a strictly feasible point).

Solution (8.3) The level sets of $f(x_1,x_2) = x_1^2 + 2x_2^2$ are ellipses. The constraints define one circle of radius 1 centred at $(1,0)$ and the line $x_2 = 1$. They intersect at the point $(1,1)$, where the objective function has the value 3.



The optimal solution is $\boldsymbol{x}^* = (1,1)^\top$ and the optimal value $p^* = 3$. The KKT conditions are

$$\begin{aligned}
(x_1-1)^2 + x_2^2 &\leq 1\\
x_2 &= 1\\
\lambda &\geq 0\\
2x_1 + 2\lambda(x_1-1) &= 0\\
4x_2 + 2\lambda x_2 + \mu &= 0\\
\lambda((x_1-1)^2 + x_2^2 - 1) &= 0.
\end{aligned}$$

At the point $(1,1)^\top$ they take the particularly simple form

$$\lambda \geq 0, 2 = 0, 4 + 2\lambda + \mu = 0.$$

These equations have no solution, so there is no way to use the Lagrange multipliers to certify optimality.