

Solutions to Part B of Problem Sheet 9

Solution (9.4) The largest eigenvalue of \mathbf{A} can be written as

$$\lambda_{\max}(\mathbf{A}) = \max_{\mathbf{v} \in \mathbb{R}^n} \frac{\mathbf{v}^\top \mathbf{A} \mathbf{v}}{\mathbf{v}^\top \mathbf{v}}.$$

Therefore, $t \geq \lambda_{\max}(\mathbf{A})$ is equivalent to the statement that for all \mathbf{v} ,

$$t \geq \frac{\mathbf{v}^\top \mathbf{A} \mathbf{v}}{\mathbf{v}^\top \mathbf{v}} \Leftrightarrow t \mathbf{v}^\top \mathbf{v} - \mathbf{v}^\top \mathbf{A} \mathbf{v} \geq 0.$$

We can write $\mathbf{v}^\top \mathbf{v} = \mathbf{v}^\top \mathbf{I} \mathbf{v}$, with the identity matrix \mathbf{I} , so that the above is equivalent to

$$t \mathbf{I} - \mathbf{A} \succeq \mathbf{0}.$$

The following semidefinite programming problem gives the largest eigenvalue:

$$\begin{aligned} & \text{minimize} && t \\ & \text{subject to} && t \mathbf{I} - \mathbf{A} \succeq \mathbf{0}. \end{aligned}$$

Solution (9.5)

- (a) A symmetric matrix can be diagonalized by orthogonal transformations, $\Sigma = \mathbf{Q}^\top \mathbf{A} \mathbf{Q}$, with \mathbf{Q} orthogonal, and Σ is diagonal with the eigenvalues of \mathbf{A} in the diagonal. The trace is invariant under similarity transformations, $\text{tr}(\mathbf{A}) = \text{tr}(\mathbf{Q}^\top \mathbf{A} \mathbf{Q})$, from which we get that the trace equals the sum of the eigenvalues.
- (b) By Part (a), the problem is formally defined as

$$\text{minimize} \quad \mathbf{I} \bullet \mathbf{X} \quad \text{subject to} \quad x_{ij} = a_{ij} \text{ for } (i, j) \in \Omega, \quad \mathbf{X} \succeq \mathbf{0}.$$

We can formulate each of the constraints in the form $\mathbf{A}_{ij} \bullet \mathbf{X} = a_{ij}$, with \mathbf{A}_{ij} the matrix with a 1 in entry (i, j) and 0 elsewhere. The dual problem would be of the form

$$\begin{aligned} & \text{maximize} && \sum_{(i,j) \in \Omega} y_{ij} a_{ij} \\ & \text{subject to} && \sum_{(i,j) \in \Omega} y_{ij} \mathbf{A}_{ij} + \mathbf{S} = \mathbf{I} \\ & && \mathbf{S} \succeq \mathbf{0}. \end{aligned}$$