Solutions to Part A of Problem Sheet 4

Solution (4.1) The data for the first linear programming problem is

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 0 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

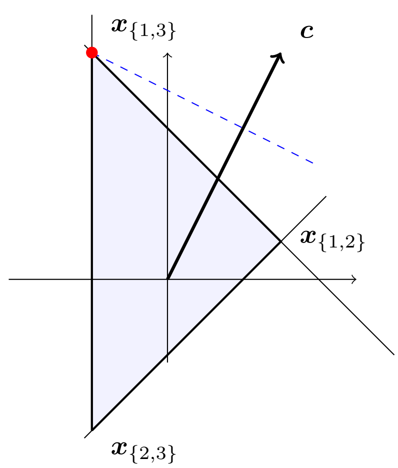
The minors are

$$\boldsymbol{A}_{\{1,2\}} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \; \boldsymbol{A}_{\{1,3\}} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \; \boldsymbol{A}_{\{2,3\}} = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}.$$

The minors are all invertible, so that the corresponding systems of linear equations $\boldsymbol{A}_I \boldsymbol{x} = \boldsymbol{b}_I$ have unique solutions, given by

$$\boldsymbol{x}_{\{1,2\}} = \begin{pmatrix} 1.5 \\ 0.5 \end{pmatrix}, \; \boldsymbol{x}_{\{1,3\}} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}, \; \boldsymbol{x}_{\{2,3\}} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}.$$

One easily verifies that each of these solutions also satisfies the remaining inequality, so that they are indeed vertices. This can also easily be seen by drawing the resulting polyhedron as intersection of halfspaces, as in the figure. The objective values $\langle \boldsymbol{c}, \boldsymbol{x} \rangle$ at



these points are

$$\langle \boldsymbol{c}, \boldsymbol{x}_{\{1,2\}} \rangle = 2.5, \; \langle \boldsymbol{c}, \boldsymbol{x}_{\{1,3\}} \rangle = 5, \; \langle \boldsymbol{c}, \boldsymbol{x}_{\{2,3\}} \rangle = -5.$$

The optimal point is therefore $\boldsymbol{x}_{\{1,3\}} = (-1, 3)^\top$. Finally, we write down the dual problem

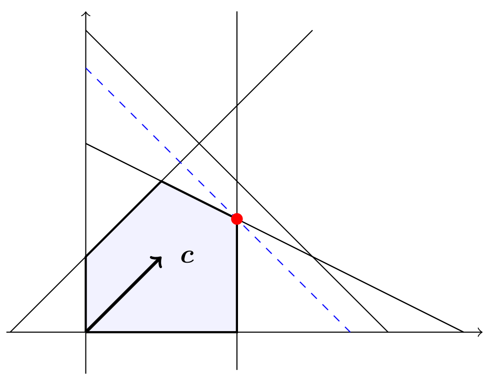
$$\text{minimize} \quad \langle \boldsymbol{b}, \boldsymbol{y} \rangle \qquad \text{subject to} \quad \boldsymbol{A}^\top \boldsymbol{y} = \boldsymbol{c}, \; \boldsymbol{y} \geq \boldsymbol{0}.$$

In our case,

$$\begin{aligned} \text{minimize} \quad & 2y_1 + y_2 + y_3 \\ \text{subject to} \quad & y_1 + y_2 - y_3 = 1 \\ & y_1 - y_2 = 2 \\ & y_i \geq 0,\ 1 \leq i \leq 3. \end{aligned}$$

For the second optimization problem, we begin by sketching the feasible set; this will make it easier to identify the vertices.

$$\boldsymbol{A} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \\ 1 & 2 \\ 1 & 0 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 2 \\ 8 \\ 10 \\ 4 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{c} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$



What we see from the sketch is that we can get rid of inequality 2 altogether, as it doesn't contribute to the polyhedron. Among the other inequalities, any two non-parallel lines intersect and therefore give rise to a non-singular minor, but there are only five of these pairs that give rise to points *in* P, and therefore define vertices. These vertices are

$$\boldsymbol{x}_{\{5,6\}} = (0,0)^\top,\ \boldsymbol{x}_{\{1,5\}} = (0,2)^\top,\ \boldsymbol{x}_{\{1,3\}} = (2,4)^\top,\ \boldsymbol{x}_{\{3,4\}} = (4,3)^\top,\ \boldsymbol{x}_{\{4,6\}} = (4,0)^\top.$$

Evaluating the objective functions on these gives the values (in the order of the vertices given above)

$$0,\ 2,\ 6,\ 7,\ 4.$$

As seen in the sketch, the optimal value is attained at $\boldsymbol{x}_{\{3,4\}} = (4,3)^\top$. Finally, the dual problem is given by

$$\begin{aligned} \text{minimize} \quad & 2y_1 + 8y_2 + 10y_3 + 4y_4 \\ \text{subject to} \quad & -1y_1 + y_2 + y_3 + y_4 - y_5 = 1 \\ & y_1 + y_2 + 2y_3 - y_6 = 1 \\ & y_i \geq 0,\ 1 \leq i \leq 6. \end{aligned}$$

Solution (4.2) Suppose that there exists a nonzero $\boldsymbol{x} \geq \boldsymbol{0}$ such that $\boldsymbol{A}\boldsymbol{x} = \sum_{i=1}^n x_i \boldsymbol{a}_i = \boldsymbol{0}$, where the $\boldsymbol{a}_i$ are the columns of $\boldsymbol{A}$. Now if there was a $\boldsymbol{0} \neq \boldsymbol{y} \in \mathbb{R}^m$ such that $\boldsymbol{A}^\top \boldsymbol{y} > 0$, then $\langle \boldsymbol{a}_i, \boldsymbol{y} \rangle > 0$ for all $1 \leq i \leq n$, so that

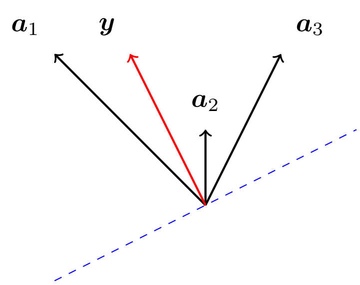
$$0 = \langle \boldsymbol{0}, \boldsymbol{y} \rangle = \sum_{i=1}^n x_i \langle \boldsymbol{a}_i, \boldsymbol{y} \rangle > 0,$$

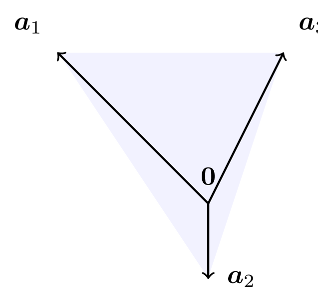
which is absurd. Therefore, such a $\boldsymbol{y}$ does not exist. Now assuming there is $\boldsymbol{y}$ with $\boldsymbol{A}^\top \boldsymbol{y} > 0$, then, for all nonzero $\boldsymbol{x} \geq \boldsymbol{0}$,

$$\langle \boldsymbol{x}, \boldsymbol{A}^\top \boldsymbol{y} \rangle = \langle \boldsymbol{A}\boldsymbol{x}, \boldsymbol{y} \rangle \neq 0,$$

which shows that $\boldsymbol{A}\boldsymbol{x} \neq \boldsymbol{0}$ for all such $\boldsymbol{x}$.

For the geometric interpretation, consider the following two diagrams.





The interpretation is that either the columns of $\boldsymbol{A}$, $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$, are all on one side of a hyperplane, or $\boldsymbol{0}$ can be written as convex combination of the $\boldsymbol{a}_i$.